South East Asian J. of Mathematics and Mathematical Sciences
Vol. 21, No. 3 (2025), pp. 139-152

DOI: 10.56827/SEAJMMS.2025.2103.9 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

GENERALIZED KOTHE-TOEPLITZ DUALS OF A NEW CLASS OF SEQUENCE SPACES

Mohd Javed and Seraj Ahemed Khan

Department of Mathematics,
Shibli National College,
Azamgarh - 276001, Uttar Pradesh, INDIA

E-mail : javed007m@gmail.com, Khanseraj86@gmail.com

(**Received:** *Sep. 05, 2025* **Accepted:** *Dec. 28, 2025* **Published:** *Dec. 30, 2025)*

Abstract: In this paper, we determine the $p\alpha$-duals of certain newly introduced difference sequence spaces, namely $\ell_\infty(u, \triangle_v^m, q), c_0(u, \triangle_v^m, q)$, and $c(u, \triangle_v^m, q)$. Furthermore, we investigate their topological properties and establish the conditions under which these spaces are perfect.

Keywords and Phrases: Sequence spaces, Köthe Toeplitz duals, α-duals, η-duals.

2020 Mathematics Subject Classification: 46A45, 46B10, 46B45.

1. Introduction and preliminaries

Assume that ω denotes the space of all sequences. Throughout this chapter, we adopt the notation $q = (q_k)$ to denote a sequence of positive real numbers. Some generalized sequence spaces, such as $\ell_\infty(q)$, $c(q)$, and $c_0(q)$, have been studied by various authors [17-19]. Ahmad and Mursaleen [1] introduced the concept of some new generalized difference sequence spaces, defined as follows:

$$\begin{aligned}
\triangle\ell_\infty(q) &= \{x \in \omega : \triangle x \in \ell_\infty(q)\}. \\
\triangle c(q) &= \{x \in \omega : \triangle x \in c(q)\}. \\
\triangle c_0(q) &= \{x \in \omega : \triangle x \in c_0(q)\}.
\end{aligned}$$

Where $\triangle x_k = x_k - x_{k+1}$. After that, Et and Başarır [16] defined these difference sequence spaces in a more general form by introducing the operator $\triangle^m$on these spaces.

$$\begin{aligned} \triangle^m \ell_\infty(q) &= \{x \in \omega : \triangle^m x \in \ell_\infty(q)\}. \\ \triangle^m c(q) &= \{x \in \omega : \triangle^m x \in c(q)\}. \\ \triangle^m c_0(q) &= \{x \in \omega : \triangle^m x \in c_0(q)\}. \end{aligned}$$

Where m is any positive integer and $\triangle^0 x_k = x_k$, $\triangle^m x_k = \triangle^{m-1} x_k - \triangle^{m-1} x_{k+1}$ and $\triangle^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}$. They also obtained the Köthe–Toeplitz duals of the space $\triangle^m \ell_\infty(q)$. In 2004 (see [11]), C. Asma Bektaş gave the β- and γ-duals of the spaces $\triangle^m \ell_\infty(q)$ and $\triangle^m c(q)$. They also gave the α-duals of $\triangle^m c_0(q)$ and $\triangle^m c(q)$. Let $v = (v_k)$ be used to denote any sequence of non-zero complex numbers. Et and Esi [15] extended the concept of difference sequence spaces to introduce the sequence spaces $\triangle_v^m(X)$ defined as below

$$\triangle_v^m(X) = \{x \in \omega : \triangle_v^m x \in X\}.$$

Where $\triangle_v^m(x_k) = \triangle_v^{m-1} x_k - \triangle_v^{m-1} x_{k+1}$ and $\triangle_v^m(x_k) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$. Bektaş, Et, and Çolak [12] defined the sequence space $\triangle_v^m(X)$ for $X = \ell_\infty, c, c_0$ and found the β- and γ-duals of these spaces. Let U denote the collection of all sequences consisting of non-zero complex numbers. That is, if $u = (u_k) \in U$ then we must have $u_k \neq 0$ for all k. Throughout this article, we will use w_k for $|u_k|^{-1}$. By taking a sequence of non-zero terms (u_k), Malkowsky [20] introduced a new class of difference sequence spaces $X(u, \triangle)$ defined as

$$X(u, \triangle) = \{x \in \omega : (u_k \triangle x_k) \in X\}.$$

Where $u \in U$ and X denotes one of the spaces ℓ_∞, c, c_0.
Asma and Çolak [6] combined the above concept with generalized sequence spaces and introduced a new class of sequence space $X(u, \triangle, q)$, defined as follows

$$X(u, \triangle, q) = \{x \in \omega : (u_k \triangle x_k) \in X(q)\}.$$

Later, Bektaş A. replaced $\triangle$ with $\triangle^2$ and introduced another class of sequence spaces $X(u, \triangle^2, q)$. Many authors [8, 13] have studied the Köthe-Toeplitz duals of such sequence spaces. The reader may refer to the articles [2, 3, 4, 5, 7, 8, 10, 14, 23], which are among the first publications in this research area, and to the recent textbooks [21] and [9] for new results on sequence spaces generated by certain triangles, as well as for fundamental theorems in functional analysis, summability theory, and sequence spaces. In 2024, Gülcan A.T. (see [22]) introduced some new sequence spaces, defined below, and also determined the α-,β-, and γ-duals of these spaces.

$$\begin{aligned} \ell_\infty(u, \triangle_v^m, q) &= \{x \in \omega : (u_k \triangle_v^m x_k) \in \ell_\infty(q)\} \\ c(u, \triangle_v^m, q) &= \{x \in \omega : (u_k \triangle_v^m x_k) \in c(q)\} \\ c_0(u, \triangle_v^m, q) &= \{x \in \omega : (u_k \triangle_v^m x_k) \in c_0(q)\} \end{aligned}$$

In the present paper, we continue our work on the spaces $X(u, \triangle_v^m, q)$ and determine the $p\alpha$, $p\beta$, and $p\gamma$-duals of these spaces. We also show that the results of Gülcan A.T. and many other researchers are special cases of our findings. Furthermore, our results generalize all previously known results on these spaces.

2. Definition of the pα-, pβ- and pγ-duals of a sequence space

The definition of $p\alpha$-, $p\beta$- and $p\gamma$- duals of sequence spaces was defined by Et [15]. Let E be a sequence space and $p > 0$, then

$$E^{p\alpha} = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k|^p < \infty, \text{ for each } x = (x_k) \in E\},$$

$$E^{p\beta} = \{a = (a_k) : \sum_{k=1}^{\infty} (a_k x_k)^p \text{ is convergent for each } x = (x_k) \in E\},$$

$$E^{p\gamma} = \{a = (a_k) : \sup_n |\sum_{k=1}^{n} (a_k x_k)^p| < \infty, \text{ for each } \ x \in X\}.$$

$E^{p\alpha}$, $E^{p\beta}$ and $E^{p\gamma}$ are called $p\alpha$-, $p\beta$- and $p\gamma$- duals of E. If we replace p with 1 in the above definitions, then we get the $\alpha-$, $\beta-$ and $\gamma-$ duals of E. If $E \subseteq F$ then $F^\eta \subseteq E^\eta$, where η is stand for $p\alpha$-, $p\beta$- and $p\gamma$.

3. Main Results

We start our discussion with the following established result, which plays a key role in developing the proof of our main results.

Theorem 3.1. *$c_0(u, \triangle_v^m, q)$, $c(u, \triangle_v^m, q)$ and $\ell_\infty(u, \triangle_v^m, q)$ are linear spaces.* [22]

Theorem 3.2. *Let $q = (q_k)$ be bounded and $M = \max(1, H = \sup_n q_n)$. Then $\ell_\infty(u, \triangle_v^m, q)$ and $c_0(u, \triangle_v^m, q)$ are linear topological spaces by g, defined by*

$$g(x) = \sup_n |u_n \triangle_v^m x_n|^{\frac{q_n}{M}}$$

Furthermore $c(u, \triangle_v^m, q)$ is paranormed by g if $\inf_n q_n > 0$. [22]

Theorem 3.3. *Let $q = (q_k)$ denote a sequence of positive real numbers. Then*

1. $[\ell_\infty(u, \triangle_v^m, q)]^{p\alpha} = D_{p\alpha}(u, q) =$
$\bigcap_{s=2}^{\infty}\{a = (a_k) : \sum_{k=1}^{\infty} |a_k|^r |v_k|^{-r} [\sum_{i=1}^{k-m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i]^r < \infty\}$,

2. $[\ell_\infty(u, \triangle_v^m, q)]^{p\alpha p\alpha} = D_{p\alpha p\alpha}(u, q) =$
$\bigcup_{s=2}^{\infty}\{a = (a_k) : \sup_{k\geq m+1} |a_k|^r |v_k|^r [\sum_{i=1}^{k-m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i]^{-r} < \infty\}$.

Proof. Let $x = (x_k)$ be any element of $\ell_\infty(u, \triangle_v^m, q)$. Then, we have

$$\begin{aligned} u_k \triangle_v^m x_k &\in \ell_\infty(q) \\ \Rightarrow \sup_k |u_k \triangle_v^m x_k|^{q_k} &< \infty \end{aligned}$$

Define an integer s as $s > \max\{1, \sup_k |u_k \triangle_v^m x_k|^{q_k}\}$. Then

$$\begin{aligned} \sup_k |u_k \triangle_v^m x_k|^{q_k} &< s \\ \Rightarrow |u_k \triangle_v^m x_k|^{q_k} &< s, \ \forall k \\ \Rightarrow |u_k||\triangle_v^m x_k| &< s^{\frac{1}{q_k}}, \ \forall k \end{aligned}$$

But $|u_k|^{-1} = w_k$, so we must have

$$|\triangle_v^m x_k| < s^{\frac{1}{q_k}} w_k \tag{3.1}$$

Again, let $a = (a_k) \in D_{p\alpha}(u, q)$. Then

$$\sum_{k=1}^{\infty} |a_k|^r |v_k|^{-r} [\sum_{i=1}^{k-m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i]^r < \infty \tag{3.2}$$

Now, for $k = 2m, 2m+1, ...$ and $s > 1$, we can write

$$\sum_{i=1}^{k-m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i \geq \sum_{i=1}^{m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i \tag{3.3}$$

Since, we can write x_k as

$$x_k = v_k^{-1}[\sum_{i=1}^{k-m} (-1)^m \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} \triangle_v^m x_i + \sum_{i=1}^{m} (-1)^{m-i} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} \triangle_v^{m-i} x_i]$$

Consider,
$\sum_{k=1}^{\infty}|a_kx_k|^r = \sum_{k=1}^{\infty}|a_k|^r|v_k|^{-r}|\sum_{i=1}^{k-m}(-1)^m \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} \triangle_v^m x_i +$
$\sum_{i=1}^{m}(-1)^{m-i} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} \triangle_v^{m-i} x_i|^r$
i.e. $\sum_{k=1}^{\infty}|a_kx_k|^r \leq \sum_{k=1}^{\infty}|a_k|^r|v_k|^{-r}\{\sum_{i=1}^{k-m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} |\triangle_v^m x_i|\}^r +$
$|a_k|^r|v_k|^{-r}\{\sum_{i=1}^{m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} |\triangle_v^{m-i} x_i|\}^r$
using (3.1), above, we get
$\sum_{k=1}^{\infty}|a_kx_k|^r < \sum_{k=1}^{\infty}|a_k|^r|v_k|^{-r}\{\sum_{i=1}^{k-m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i\}^r +$
$|a_k|^r|v_k|^{-r}\{\sum_{i=1}^{m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i\}^r < \infty$, using (3.2) and (3.3)
i.e., $\sum_{k=1}^{\infty}|a_kx_k|^r < \infty$ for all $(x_k) \in \ell_\infty(u, \triangle_v^m, q)$. This means that (a_k) is an element of $[\ell_\infty(u, \triangle_v^m, q)]^{p\alpha}$, but (a_k) is an element of $D_{p\alpha}(u, q)$. Hence, we must have

$$D_{p\alpha}(u, q) \subseteq [\ell_\infty(u, \triangle_v^m, q)]^{p\alpha}$$

Now, to complete the proof of the first part of Theorem 3.3, we only have to prove that $[\ell_\infty(u, \triangle_v^m, q)]^{p\alpha} \subseteq D_{p\alpha}(u, q)$. For this, we let $(a_k) \notin D_{p\alpha}(u, q)$. Then, there exists some integer $s > 1$ such that

$$\sum_{k=1}^{\infty}|a_k|^r|v_k|^{-r}\left\{\sum_{i=1}^{k-m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i\right\}^r = \infty$$

Using the above expression, we can define a sequence, say $x = (x_k)$, where $x_k = v_k^{-r}\sum_{i=1}^{k-m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i$, which can easily be verified to be an element of $\ell_\infty(u, \triangle_v^m, q)$. Then, we have

$$\sum_{k=1}^{\infty}|a_kx_k|^r = \infty$$

This means $(a_k) \notin [\ell_\infty(u, \triangle_v^m, q)]^{p\alpha}$. Hence $[\ell_\infty(u, \triangle_v^m, q)]^{p\alpha} = D_{p\alpha}(u, q)$.This completes the first part of Theorem 3.3. After this, we are ready to prove the second part of Theorem 3.3.
(ii) Let $a = (a_k) \in D_{p\alpha p\alpha}(u, q)$ and $x = (x_k) \in [\ell_\infty(u, \triangle_v^m, q)]^{p\alpha}$. Then, we have

$$\sup_{k\geq m+1}|a_k|^r|v_k|^r[\sum_{i=1}^{k-m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i]^{-r} < \infty \tag{3.4}$$

for some $s > 1$ and

$$\sum_{k=1}^{\infty} |x_k|^r |v_k|^{-r} [\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{1}{q_i}} w_i]^r < \infty \tag{3.5}$$

for all $s > 1$. Now, we consider

$$\begin{aligned} \sum_{k=m+1}^{\infty} |a_k x_k|^r &= \sum_{k=m+1}^{\infty} |a_k|^r |v_k|^r [\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{1}{q_i}} w_i]^{-r} \\ &\times |x_k|^r |v_k|^{-r} [\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{1}{q_i}} w_i]^r \\ &= \sup_{k \geq m+1} |a_k|^r |v_k|^r [\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{1}{q_i}} w_i]^{-r} \\ &\times \sum_{k=m+1}^{\infty} |x_k|^r |v_k|^{-r} [\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{1}{q_i}} w_i]^r \\ &< \infty, \text{ In view of (3.4) and (3.5)} \end{aligned}$$

i.e., $\sum_{k=m+1}^{\infty} |a_k x_k|^r < \infty$ for all $x = (x_k) \in [\ell_\infty(u, \triangle_v^m, q)]^{p\alpha}$. By the definition of $p\alpha$-duals, we can say that $a = (a_k)$ is an element of $[\ell_\infty(u, \triangle_v^m, q)]^{p\alpha p\alpha}$, but $a = (a_k)$ is any arbitrary element of $D_{p\alpha p\alpha}(u, q)$. Therefore, we must have

$$D_{p\alpha p\alpha}(u, q) \subseteq [\ell_\infty(u, \triangle_v^m, q)]^{p\alpha p\alpha}$$

Now, to complete proof of the second part of Theorem 3.3, we only need to prove that
$[\ell_\infty(u, \triangle_v^m, q)]^{p\alpha p\alpha} \subseteq D_{p\alpha p\alpha}(u, q)$. To do this, let us suppose that $a = (a_k) \in [\ell_\infty(u, \triangle_v^m, q)]^{p\alpha p\alpha}$. Then, we have

$$\sum_{k=1}^{\infty} |a_k x_k|^r < \infty, \text{ for all } x = (x_k) \in [\ell_\infty(u, \triangle_v^m, q)]^{p\alpha}$$

But if $x = (x_k) \in [\ell_\infty(u, \triangle_v^m, q)]^{p\alpha}$. Then, by the proof of the first part of Theorem 3.3, we must have

$$\sum_{k=1}^{\infty} |x_k|^r |v_k|^{-r} [\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{1}{q_i}} w_i]^r < \infty \tag{3.6}$$

for all $s \geq 2$. Further, as $\sum_{k=1}^{\infty} |a_k x_k|^r < \infty$. So, we have

$$\begin{aligned} \sum_{k=1}^{\infty} |a_k|^r |v_k|^r [\sum_{i=1}^{k-m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i]^{-r} & \\ \times |x_k|^r |v_k|^{-r} [\sum_{i=1}^{k-m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i]^{r} &< \infty \\ \Rightarrow |a_k|^r |v_k|^r [\sum_{i=1}^{k-m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i]^{-r} &< \infty, \ \forall\ n \text{ , By using (3.6)} \\ \Rightarrow \sup_{k \geq m+1} |a_k|^r |v_k|^r [\sum_{i=1}^{k-m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i]^{-r} &< \infty \end{aligned}$$

This means $a = (a_k) \in D_{p\alpha p\alpha}(u, q)$ for all $a = (a_k) \in [\ell_\infty(u, \triangle_v^m, q)]^{p\alpha p\alpha}$. Hence,

$$[\ell_\infty(u, \triangle_v^m, q)]^{p\alpha p\alpha} \subseteq D_{p\alpha p\alpha}(u, q)$$

This completes the proof of second part of Theorem 3.3. One more thing we can add is that since $[\ell_\infty(u, \triangle_v^m, q)]^{p\alpha p\alpha} \neq \ell_\infty(u, \triangle_v^m, q)$, the space $\ell_\infty(u, \triangle_v^m, q)$is not perfect.

Corollary 3.1. *If we put $r = 1$ in Theorem 3.3, then we get the α-dual of the space $\ell_\infty(u, \triangle_v^m, q)$.*

1. $[\ell_\infty(u, \triangle_v^m, q)]^{\alpha} = D_\alpha(u, q) =$
   $\bigcap_{s=2}^{\infty} \{a = (a_k) : \sum_{k=1}^{\infty} |a_k||v_k|^{-1} [\sum_{i=1}^{k-m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i] < \infty\}$,
2. $[\ell_\infty(u, \triangle_v^m, q)]^{\alpha\alpha} = D_{\alpha\alpha}(u, q) =$
   $\bigcup_{s=2}^{\infty} \{a = (a_k) : \sup_{k \geq m+1} |a_k||v_k| [\sum_{i=1}^{k-m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i]^{-1} < \infty\}$.

Corollary 3.2. *If we put $v_k = 1, \forall\ k$ in Theorem 3.3, then we get the $p\alpha$-dual of the space $\ell_\infty(u, \triangle^m, q)$.*

1. $[\ell_\infty(u, \triangle^m, q)]^{p\alpha} =$
   $\bigcap_{s=2}^{\infty} \{a = (a_k) : \sum_{k=1}^{\infty} |a_k|^r [\sum_{i=1}^{k-m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i]^{r} < \infty\}$,
2. $[\ell_\infty(u, \triangle_v^m, q)]^{p\alpha p\alpha} =$
   $\bigcup_{s=2}^{\infty} \{a = (a_k) : \sup_{k \geq m+1} |a_k|^r [\sum_{i=1}^{k-m} \begin{pmatrix} k-i-1 \\ m-1 \end{pmatrix} s^{\frac{1}{q_i}} w_i]^{-r} < \infty\}$.

4. First and second pα-duals of the space $c_0(u, \triangle_v^m, q)$

In the next theorem, we will find the first and second $p\alpha$-duals of the space $c_0(u, \triangle_v^m, q)$ and prove that this space is also not perfect.

Theorem 4.1. *Let $q = (q_k)$ denote a sequence of strictly positive real numbers. Then, we have*

1. $[c_0(u, \triangle_v^m, q)]^{p\alpha} = M_{p\alpha}(u, q)$, *where*

$$M_{p\alpha}(u,q) = \bigcup_{s=2}^{\infty}\{a = (a_k) : \sum_{k=1}^{\infty} |a_k|^r |v_k|^{-r} \left[\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{-1}{q_i}} w_i\right]^r < \infty\}$$

2. $[c_0(u, \triangle_v^m, q)]^{p\alpha p\alpha} = M_{p\alpha p\alpha}(u, q)$, *where*

$$M_{p\alpha p\alpha}(u,q) = \bigcap_{s=2}^{\infty}\{a \in \omega : \sup_{k \geq m+1} |a_k|^r |v_k|^r \left[\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{-1}{q_i}} w_i\right]^{-r} < \infty\}.$$

Proof. (1) Let $x = (x_k)$ be an arbitrary element of $c_0(u, \triangle_v^m, q)$. Then

$$\begin{aligned} u_k \triangle_v^m x_k &\in c_0(q) \\ \Rightarrow |u_k \triangle_v^m x_k|^{q_k} &\in c_0 \\ i.e., \ |u_k \triangle_v^m x_k|^{q_k} &\to 0 \text{ as } k \to \infty. \end{aligned}$$

This means there must exists a positive integer $s > 1$ such that

$$\begin{aligned} |u_k \triangle_v^m x_k|^{q_k} &< \frac{1}{s} \\ i.e., \ |u_k \triangle_v^m x_k| &< s^{\frac{-1}{q_k}}. \end{aligned}$$

$$\Longrightarrow |\triangle_v^m x_k| < s^{\frac{-1}{q_k}} w_k. \tag{4.1}$$

Again, let $a = (a_k) \in M_{p\alpha}(u, q)$. Then

$$\sum_{k=1}^{\infty} |a_k|^r |v_k|^{-r} \left[\sum_{i=1}^{k-m} \binom{k-i-1}{m-i} s^{\frac{-1}{q_i}} w_i\right]^r < \infty \tag{4.2}$$

for some $s > 1$. Since

$$\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{-1}{q_i}} w_i \geq \sum_{i=1}^{m} \binom{k-i-1}{m-1} s^{\frac{-1}{q_i}} w_i. \tag{4.3}$$

Then

$$\begin{aligned}
\sum_{i=1}^{\infty}|a_i x_i|^r &= \sum_{i=1}^{\infty}|a_i|^r|v_i|^{-r}|\sum_{i=1}^{k-m}(-1)^m\binom{k-i-1}{m-1}\triangle_v^m x_i \\
&+ \sum_{i=1}^{m}(-1)^{m-i}\binom{k-i-1}{m-1}\triangle_v^{m-i}x_i|^r \\
&\le \sum_{i=1}^{\infty}|a_i|^r|v_i|^{-r}\left[\sum_{i=1}^{k-m}(-1)^m\binom{k-i-1}{m-1}|\triangle_v^m x_i|\right]^r \\
&+ \sum_{i=1}^{\infty}|a_i|^r|v_i|^{-r}\left[\sum_{i=1}^{m}(-1)^{m-i}\binom{k-i-1}{m-1}|\triangle_v^{m-i}x_i|\right]^r \\
&\le \sum_{i=1}^{\infty}|a_i|^r|v_i|^{-r}\left[\sum_{i=1}^{k-m}(-1)^m\binom{k-i-1}{m-1}s^{\frac{-1}{q_i}}w_i\right]^r \\
&+ \sum_{i=1}^{\infty}|a_i|^r|v_i|^{-r}\left[\sum_{i=1}^{m}(-1)^{m-i}\binom{k-i-1}{m-1}s^{\frac{-1}{q_i}}w_i\right]^r, \text{ see (4.1)} \\
&< \infty, \text{ see (4.2) and (4.3)}
\end{aligned}$$

i.e., $\sum_{i=1}^{\infty}|a_i x_i|^r < \infty$, for all $x = (x_k) \in c_0(u, \triangle_v^m, q)$.

$$\Rightarrow a = (a_k) \in [c_0(u, \triangle_v^m, q)]^{p\alpha}.$$

Hence,

$$M_{p\alpha}(u, q) \subseteq [c_0(u, \triangle_v^m, q)]^{p\alpha}. \tag{4.4}$$

To complete the proof of first part of Theorem 4.1, we only need to prove that $[c_0(u, \triangle_v^m, q)]^{p\alpha} \subseteq M_{p\alpha}(u, q)$. For this, let us suppose that $a = (a_k) \in [c_0(u, \triangle_v^m, q)]^{p\alpha}$. Then, we must have

$$\sum_{i=1}^{\infty}|a_i x_i|^r < \infty, \text{ for all } x = (x_k) \in c_0(u, \triangle_v^m, q). \tag{4.5}$$

Define a sequence as follows:

$$x_k = v_k^{-1}\sum_{i=1}^{k-m}\binom{k-i-1}{m-1}s^{\frac{-1}{q_i}}w_i. \tag{4.6}$$

We can easily verify that $x=(x_k)$ is an element of $c_0(u,\triangle_v^m,q)$. In view of (4.5) and (4.6) , we have

$$\sum_{k=1}^{\infty}|a_k|^r|v_k|^{-r}\left[\sum_{i=1}^{k-m}\binom{k-i-1}{m-1}s^{\frac{-1}{q_i}}w_i\right]^r<\infty.$$

$$\Longrightarrow a=(a_k)\in M_{p\alpha}(u,q).$$

$$\Longrightarrow [c_0(u,\triangle_v^m,q)]^{p\alpha}\subseteq M_{p\alpha}(u,q). \tag{4.7}$$

From (4.4) and (4.7), we have

$$[c_0(u,\triangle_v^m,q)]^{p\alpha}=M_{p\alpha}(u,q).$$

This completes the proof of first part of Theorem 4.1. Now, we will prove the second part and then verify that this space is not perfect.

Proof. (2) Let $a=(a_k)\in[c_0(u,\triangle_v^m,q)]^{p\alpha p\alpha}$. Then, we have

$$\sum_{k=m+1}^{\infty}|a_kx_k|^r<\infty, \tag{4.8}$$

for all $x=x_k\in[c_0(u,\triangle_v^m,q)]^{p\alpha}$. Now, using the proof of the first part of Theorem 4.1, we can write

$$\sum_{k=1}^{\infty}|x_k|^r|v_k|^{-r}\left[\sum_{i=1}^{k-m}\binom{k-i-1}{m-1}s^{\frac{-1}{q_i}}w_i\right]^r<\infty \tag{4.9}$$

for some $s\geq 2$. We can rewrite (4.8), as follows:

$$\sum_{k=m+1}^{\infty}|a_k|^r|v_k|^r\left[\sum_{i=1}^{k-m}\binom{k-i-1}{m-1}s^{\frac{-1}{q_i}}w_i\right]^{-r}|x_k|^r|v_k|^{-r}$$
$$\times\left[\sum_{i=1}^{k-m}\binom{k-i-1}{m-1}s^{\frac{-1}{q_i}}w_i\right]^r<\infty$$

$$\Longrightarrow \sup_{k\geq m+1}|a_k|^r|v_k|^r\left[\sum_{i=1}^{k-m}\binom{k-i-1}{m-1}s^{\frac{-1}{q_i}}w_i\right]^{-r}\sum_{k=m+1}^{\infty}|x_k|^r|v_k|^{-r}\times$$
$$\left[\sum_{i=1}^{k-m}\binom{k-i-1}{m-1}s^{\frac{-1}{q_i}}w_i\right]^r<\infty.$$

In view of (4.9), this gives

$$\sup_{k\geq m+1} |a_k|^r |v_k|^r \left[\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{-1}{q_i}} w_i\right]^{-r} < \infty.$$

This means $a = (a_k) \in M_{p\alpha p\alpha}(u, q)$. Hence

$$[c_0(u, \triangle_v^m, q)]^{p\alpha p\alpha} \subseteq M_{p\alpha p\alpha}(u, q). \tag{4.10}$$

To complete the proof of Theorem 4.1, we claim that $M_{p\alpha p\alpha}(u, q) \subseteq [c_0(u, \triangle_v^m, q)]^{p\alpha p\alpha}$. To do so, let us suppose that $a = (a_k)$ is an arbitrary element of $M_{p\alpha p\alpha}(u, q)$. Then, we have

$$\sup_{k\geq m+1} |a_k|^r |v_k|^r \left[\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{-1}{q_i}} w_i\right]^{-r} < \infty. \tag{4.11}$$

Let $x = (x_k)$ be any element of $[c_0(u, \triangle_v^m, q)]^{p\alpha}$. Using the result from the proof of the first part of Theorem 4.1, we can write

$$\sum_{k=1}^{\infty} |x_k|^r |v_k|^{-r} \left[\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{-1}{q_i}} w_i\right]^{r} < \infty. \tag{4.12}$$

Consider

$$\begin{aligned}
\sum_{k=m+1}^{\infty} |a_k x_k|^r &< \sum_{k=m+1}^{\infty} |a_k|^r |v_k|^r \left[\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{-1}{q_i}} w_i\right]^{-r} \\
&\times |x_k|^r |v_k|^{-r} \left[\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{-1}{q_i}} w_i\right]^{r} \\
&\leq \sup_{k=m+1} |a_k|^r |v_k|^r \left[\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{-1}{q_i}} w_i\right]^{-r} \\
&\times \sum_{k=m+1}^{\infty} |x_k|^r |v_k|^{-r} \left[\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{-1}{q_i}} w_i\right]^{r} \\
&< \infty
\end{aligned}$$

i.e., $\sum_{k=m+1}^{\infty} |a_k x_k|^r < \infty$, for all $x = (x_k) \in [c_0(u, \triangle_v^m, q)]^{p\alpha}$.

$$\Longrightarrow a = (a_k) \in [c_0(u, \triangle_v^m, q)]^{p\alpha p\alpha}.$$

Hence,

$$[c_0(u, \triangle_v^m, q)]^{p\alpha p\alpha} = M_{p\alpha p\alpha}(u, q).$$

Corollary 4.1. *If we replace r with 1 in Theorem 4.1, then we obtain the α-dual of the space $c_0(u, \triangle_v^m, q)$ [22]*

1. $[c_0(u, \triangle_v^m, q)]^{\alpha} = M_{\alpha}(u, q)$, *where*

$$M_{\alpha}(u, q) = \bigcup_{s=2}^{\infty}\{a = (a_k) : \sum_{k=1}^{\infty} |a_k||v_k|^{-1} \left[\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{-1}{q_i}} w_i\right] < \infty\}.$$

2. $[c_0(u, \triangle_v^m, q)]^{\alpha\alpha} = M_{\alpha\alpha}(u, q)$, *where*

$$M_{\alpha\alpha}(u, q) = \bigcap_{s=2}^{\infty}\{a \in \omega : \sup_{k \geq m+1} |a_k||v_k| \left[\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{-1}{q_i}} w_i\right]^{-1} < \infty\}.$$

This is exactly the same result as in Theorem 2.4 (see [22]). This means that, by Theorem 4.1, we have generalized the previously existing results.

Corollary 4.2. *If we take the sequence $v = (v_k)$ as $v_k = 1$ for all k, then we get the $p\alpha$-duals of the space $c_0(u, \triangle^m, q)$.*

1. $[c_0(u, \triangle^m, q)]^{p\alpha} = M_{p\alpha}(u, q)$, *where*

$$M_{p\alpha}(u, q) = \bigcup_{s=2}^{\infty}\{a = (a_k) : \sum_{k=1}^{\infty} |a_k|^r \left[\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{-1}{q_i}} w_i\right]^r < \infty\}.$$

2. $[c_0(u, \triangle^m, q)]^{p\alpha p\alpha} = M_{p\alpha p\alpha}(u, q)$, *where*

$$M_{p\alpha p\alpha}(u, q) = \bigcap_{s=2}^{\infty}\{a \in \omega : \sup_{k \geq m+1} |a_k|^r \left[\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{\frac{-1}{q_i}} w_i\right]^{-r} < \infty\}.$$

Corollary 4.3. [6] *If we put $m = r = 1$ and $v_k = 1$ for all k in the Theorem 3.3(1), and Theorem 4.1(1), then*

1. $[\ell_\infty(u, \triangle, q)]^{\alpha} = D_{\alpha}(u, q) = \bigcap_{s=2}^{\infty}\{a = (a_k) : \sum_{k=1}^{\infty} |a_k| \sum_{i=1}^{k-1} s^{\frac{1}{q_i}} w_i < \infty\}$

2. $[c_0(u, \triangle, q)]^{\alpha} = M_{\alpha}(u, q)$, *where*

$$M_{\alpha}(u, q) = \bigcup_{s=2}^{\infty}\{a = (a_k) : \sum_{k=1}^{\infty} |a_k| \sum_{i=1}^{k-1} s^{\frac{-1}{q_i}} w_i < \infty\}.$$

Acknowledgment:

The authors would like to express their sincere thanks to the anonymous referee for careful reading of the article and useful suggestions.

References

[1] Ahmed , Z. U., Mursaleen, M., Köthe-Toeplitz duals of some new sequence spaces and their matrix maps, Publ. Inst. Math. (Beograd), 42(56) (1987), 57-61.

[2] Altay B., Başar F., On the paranormed Riesz sequence spaces of non-absolute type, Southeast Asian Bull. Math., 26 (5) (2002), 701–715.

[3] Altay B., Başar F., Some paranormed sequence spaces of non-absolute type derived by weighted mean, J. Math. Anal. Appl., 319(2) (2006), 494–508.

[4] Altay B., Başar F., Some paranormed Riesz sequence spaces of non-absolute type, Southeast Asian Bull. Math. 30 (5) (2006), 591–608.

[5] Altay B., Başar F., Generalisation of the sequence space $\ell(p)$ derived by weighted mean, J. Math. Anal. Appl., 330 (2007), 174–185.

[6] Asma, C., Colak, R. , On Köthe-Toplitz duals of some generalized sets of difference sequence space, Demonstr. Math. Vol. III, No 4, 2000.

[7] Aydın C., Başar F., Some new paranormed sequence spaces, Inform. Sci., 160 (2004), 27–40.

[8] Barlak D., Bektaş Ç. A., Duals of generalized Orlicz Hilbert sequence spaces and matrix transformations, Filomat, 31(27) (2023), 9089-9102.

[9] Başar F., Summability Theory and its Applications, 2nd ed., CRC Press/Taylor & Francis Group, Boca Raton London New York, 2022.

[10] Başar F., Yeşilkayagil M., A survey for paranormed sequence spaces generated by infinite matrices, TWMS J. Pure Appl. Math., 10 (1) (2019), 3–38.

[11] Bektaş, Ç. A., Absolute and Ordinary Köthe-Toeplitz Duals of Some Generalized Sets of Difference Sequences, Hokkaido Math. J., 33(1) (2004), 207-214.

[12] Bektaş, Ç. A., Et, M., Çolak, R., Generalized Difference Sequence Spaces and Their Dual Spaces, J. Math. Anal. Appl., 292 (2004), 423-432.

[13] Bektaş, Ç. A., On Some Generalized Difference Sequence Spaces and Köthe Toeplitz Duals, IeJEMTA, 7 (2009), 61-72.

[14] Çapan H., Başar F., Domain of the double band matrix defined by Fibonacci numbers in the Maddox's space $\ell(p)$, Electron. J. Math. Anal. Appl., 3 (2) (2015), 31–45.

[15] Et, M., Esi, A., On köthe-Toeplitz duals of generalized difference sequence spaces, Bull. Malays. Math. Sci. Soc., 23(2) (2000), 25-32.

[16] Et, M., Başarır, M., On Some New Generalized Difference Sequence Spaces, Period. Math. Hungar., 35(3) (1997), 169-175.

[17] Lascarides, C. G. and Maddox, I. J., Matrix transformations between some classes of sequences, Proc. Cambridge Phil. Soc., 68 (1970), 99-104.

[18] Lascarides, C. G, A study of certain sequence spaces of Maddox and a generalization of a theorem of Iyer, Pacific J. Math., 38 (1971), 487-500.

[19] Maddox, I. J., Spaces of strongly summable sequences, Quart. J. Math. 18(2) (1967), 345-355.

[20] Malkowsky, E., A note on the Köthe–Toeplitz duals of generalized sets of bounded and convergent difference sequences, J. Anal., 4 (1996), 81-91.

[21] Mursaleen M., Başar F., Sequence Spaces: Topics in Modern Summability Theory, Series: Mathematics and Its Applications, CRC Press/Taylor, Francis Group, Boca Raton London New York, 2020.

[22] Turan, G. A., New generalized difference sequence spaces and their Kothe-Toeplitz duals, Middle East J. Sci., 10(2) (2024), 144-150.

[23] Uçar E., Başar F., Some geometric properties of the domain of the double band matrix defined by Fibonacci numbers in the sequence space $\ell(p)$, AIP Conf. Proc., 1611 (2014), 316–324.